On the gravitational moments of a Dirac particle

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Abstract

We consider the classical theory of the Dirac massive particle in the Riemann-Cartan spacetime. We demonstrate that the translational and the Lorentz gravitational moments, obtained by means of the Gordon type decompositions of the canonical energy-momentum and spin currents, are consistently coupled to torsion and curvature, as expected.

I. INTRODUCTION

Riemann-Cartan geometry arises naturally in the gauge theory of the Poincaré group, see e.g. [1,2]. One can interpret spacetime coframe ϑ^α and local Lorentz connection $\Gamma_\alpha{}^\beta$ as the gravitational potentials related to the translation group and the Lorentz group, respectively. The two-forms of torsion and curvature, $T^\alpha := d\vartheta^\alpha + \Gamma_\beta{}^\alpha\wedge\vartheta^\beta$, $R_\alpha{}^\beta := d\Gamma_\alpha{}^\beta + \Gamma_\gamma{}^\beta\wedge\Gamma_\alpha{}^\gamma$, represent the corresponding gauge field strengths. Besides, the Riemann-Cartan spacetime carries a metric $g_{\alpha\beta}$ which is covariantly constant: $Dg_{\alpha\beta} = 0$. Usually one chooses $g_{\alpha\beta} = o_{\alpha\beta}$, the flat Minkowski metric, thus restricting oneself to orthonormal frames and coframes.

General dynamical scheme of the Poincaré gauge theory is well established. The Noether currents of matter fields, Σ_α (the canonical energy-momentum three-form) and $\tau_{\alpha\beta}$ (the canonical spin three-form), are coupled to the translational ϑ^α and the Lorentz $\Gamma^{\alpha\beta}$ gauge potentials, respectively. These currents are thus representing the two types of *gravitational charges* of a matter source.

Recently, the Dirac electron theory has been analyzed in *flat Minkowski spacetime* [3], and the structure of the canonical energy-momentum and spin currents was studied in detail. It was shown, developing analogy with electrodynamics, that a Dirac particle is naturally characterized by two gravitational moments. In simple physical terms, one can describe them as the Ampére type ring currents induced by the two gravitational charges via spin of a particle. Here we consider the Dirac theory with the gravitational field "switched on". We demonstrate the consistency of the coupling of the gravitational moments to the torsion and curvature.

It is worthwhile to mention that gravitational moments of a Dirac particle were discussed also by Kobzarev and Okun [4] and Khriplovich [5] in the framework of Einstein's general relativity theory, whereas in [6,7] gravitational moments of higher spins and in lower dimensions were studied.

In this paper, we are using the basic conventions and notations of Bjorken and Drell. In particular, the constant Minkowski metric is $o_{\alpha\beta} = \mathrm{diag}(+1, -1, -1, -1)$, and we choose the Dirac matrices γ^α in the standard form of [10]. In the exterior algebra on a spacetime, $\wedge, \rfloor$ are exterior and interior products, respectively, * is the Hodge star operator. Local orthonormal frame e_α is dual to a coframe ϑ^α one-form: $e_\alpha \rfloor \vartheta^\beta = \delta^\beta_\alpha$. Finally, starting from the volume one-form $\eta := {}^*1$, one defines Trautman's η-basis as usual by $\eta_\alpha := e_\alpha \rfloor \eta = {}^*\vartheta_\alpha$, $\eta_{\alpha\beta} := e_\beta \rfloor \eta_\alpha = {}^*(\vartheta_\alpha \wedge \vartheta_\beta)$ etc.

II. FERMIONS IN THE RIEMANN-CARTAN SPACETIME

For the description of classical Dirac particle of mass m, we will use the formalism of Clifford algebra–valued exterior forms [1]. The following matrix-valued one- or three-forms are basic objects in this approach:

$$\gamma := \gamma_\alpha \, \vartheta^\alpha \, , \qquad {}^*\gamma = \gamma^\alpha \, \eta_\alpha \, . \tag{2.1}$$

Exterior product yields a two-form:

$$\widehat{\sigma} := \frac{i}{2}\,\gamma\wedge\gamma = \frac{1}{2}\widehat{\sigma}_{\alpha\beta}\,\vartheta^\alpha\wedge\vartheta^\beta. \tag{2.2}$$

The coefficients $\widehat{\sigma}_{\alpha\beta} := i\gamma_{[\alpha}\gamma_{\beta]}$ generate infinitesimal Lorentz transformations of spinor fields. Defining another constant matrix, $\gamma_5 := -i\,\gamma^{\hat 0}\gamma^{\hat 1}\gamma^{\hat 2}\gamma^{\hat 3}$, it is straightforward to prove the fundamental identities for the Clifford algebra-valued objects:

$$\widehat{\sigma}_{\alpha\beta}\,{}^*\widehat{\sigma} = \eta_{\alpha\beta} - i\,\gamma_5\,\vartheta_\alpha\wedge\vartheta_\beta - 2i\,\vartheta_{[\alpha}\wedge e_{\beta]}\rfloor{}^*\widehat{\sigma}, \tag{2.3}$$

$${}^*\widehat{\sigma}\,\widehat{\sigma}_{\alpha\beta} = \eta_{\alpha\beta} - i\,\gamma_5\,\vartheta_\alpha\wedge\vartheta_\beta + 2i\,\vartheta_{[\alpha}\wedge e_{\beta]}\rfloor{}^*\widehat{\sigma}. \tag{2.4}$$

The Lagrangian four-form of a Dirac field Ψ is given by

$$L_{\rm D} = \frac{i}{2}\hbar\left\{\overline{\Psi}\,{}^*\gamma\wedge D\Psi + \overline{D\Psi}\wedge{}^*\gamma\,\Psi\right\} + {}^*mc\,\overline{\Psi}\Psi\,. \tag{2.5}$$

Dirac fields are local sections of the spinor $SO(1,3)$-bundle associated with the principal bundle of orthonormal frames. Hence, the spinor covariant derivative in the Riemann-Cartan spacetime is defined by

$$D\Psi := d\Psi + \frac{i}{4}\Gamma^{\alpha\beta}\wedge\widehat{\sigma}_{\alpha\beta}\,\Psi, \qquad \overline{D\Psi} = d\overline{\Psi} - \frac{i}{4}\Gamma^{\alpha\beta}\wedge\overline{\Psi}\widehat{\sigma}_{\alpha\beta}\,. \tag{2.6}$$

The Dirac field equation, which arises from the Lagrangian (2.5), reads

$$i\hbar{}^*\gamma\wedge(D\,\Psi - \tfrac{1}{2}T\,\Psi) + {}^*mc\,\Psi = 0, \tag{2.7}$$

$$i\hbar\left(\overline{D\Psi} - \tfrac{1}{2}T\,\overline{\Psi}\right)\wedge{}^*\gamma + {}^*mc\,\overline{\Psi} = 0\,, \tag{2.8}$$

where $T := e_\alpha\rfloor T^\alpha$ is the torsion trace one-form.

The standard Lagrange–Noether machinery in the gauge gravity, see e.g. [2], provides a general definition of the energy-momentum and spin currents in non-flat spacetime (accounting also for the possibility of non-minimal coupling and "Pauli-type" terms):

$$\Sigma_\alpha := e_\alpha\rfloor L - (e_\alpha\rfloor D\Psi^A)\wedge\frac{\partial L}{\partial D\Psi^A} - (e_\alpha\rfloor\Psi^A)\wedge\frac{\partial L}{\partial\Psi^A}$$
$$+D\frac{\partial L}{\partial T^\alpha} - (e_\alpha\rfloor T^\beta)\wedge\frac{\partial L}{\partial T^\beta} - (e_\alpha\rfloor R_\beta{}^\gamma)\wedge\frac{\partial L}{\partial R_\beta{}^\gamma}, \tag{2.9}$$
$$\tau_{\alpha\beta} := (\ell_{\alpha\beta}{}^A_B\,\Psi^B)\wedge\frac{\partial L}{\partial D\Psi^A} + \vartheta_{[\alpha}\wedge\frac{\partial L}{\partial T^{\beta]}} + D\,\frac{\partial L}{\partial R^{\alpha\beta}}\,. \tag{2.10}$$

Here Ψ^A is a set of arbitrary matter fields, with $\ell_{\alpha\beta}$ denoting the corresponding Lorentz group generators.

The first and second Noether theorems yield two covariant conservation laws which are fullfilled on the classical matter field equations:

$$D\Sigma_\alpha = 0, \tag{2.11}$$

$$D\tau_{\alpha\beta} + \vartheta_{[\alpha}\wedge\Sigma_{\beta]} = 0. \tag{2.12}$$

In case of the Dirac theory under consideration, matter is described by the pair of independent four-spinors $\Psi^A = \{\Psi, \overline{\Psi}\}$, and the canonical gravitational currents are straightforwardly obtained after substituting (2.5) into (2.9)-(2.10) [hereafter $D_\alpha := e_\alpha\rfloor D$]:

$$\Sigma_\alpha = \frac{i\hbar}{2}\left(\overline{\Psi}{}^*\gamma D_\alpha\Psi - D_\alpha\overline{\Psi}{}^*\gamma\Psi\right), \tag{2.13}$$
$$\tau_{\alpha\beta} = \frac{\hbar}{4}\,\vartheta_\alpha\wedge\vartheta_\beta\wedge\overline{\Psi}\gamma\gamma_5\Psi. \tag{2.14}$$

III. GRAVITATIONAL MOMENTS OF A DIRAC PARTICLE

In a theory invariant with respect to some *internal* gauge group G, the *generalized moment* is a $\mathcal{G}$-valued two-form such that its exterior differential produces the polarizational part of the related Noether current, with $\mathcal{G}$ denoting the Lie algebra of G. On the Lagrangian level, the moment couples directly to the gauge field strength. To be specific, in the Dirac–Yang-Mills theory the Noether "isospin" current $J_{_K} = -ie\overline{\Psi}\tau_{_K}\Psi$ couples to the gauge potential one-form A^K, with e denoting the coupling constant, and $\tau_{_K}$ being generators of the gauge group. The Gordon decomposition [3] reveals

a nontrivial substructure of this coupling by relating the polarization moment two-form $P_{_K} = -i\frac{e\hbar}{2mc}\overline{\Psi}\tau_{_K}{}^*\widehat{\sigma}\Psi$ directly to the gauge field strength F^K. One may expect similar results to hold when passing from internal symmetry groups to spacetime symmetries. Geometrically this means a departure from flat Minkowski spacetime by "switching on" gravity.

Technically, we can proceed along the same lines as for the Dirac–Yang-Mills theory [3]. We use (2.7)-(2.8) in order to express Ψ and $\overline{\Psi}$ in terms of the differentials, and then substitute them back into the Dirac equation and into the Noether currents (2.13)-(2.14). After some algebra, we find the squared Dirac equation in the form

$$(D\,{}^*D + 2\,S\wedge D + X)\,\Psi = 0, \tag{3.1}$$

where the three-form S and the four-form X read, respectively,

$$S := -\frac{i}{2}\,(D^*\widehat{\sigma} + T\wedge{}^*\widehat{\sigma}), \tag{3.2}$$

$$X := {}^*\left(\frac{mc}{\hbar}\right)^2 + \frac{1}{4}\,{}^*\widehat{\sigma}\wedge R_{\alpha\beta}\,\widehat{\sigma}^{\alpha\beta}$$
$$+\frac{1}{2}\Big(-d\,{}^*T + i\,{}^*\widehat{\sigma}\wedge dT - \frac{1}{2}\,T\wedge{}^*T + i\,(D^*\widehat{\sigma})\wedge T\Big). \tag{3.3}$$

Here we have used the Ricci identity for the spinor covariant derivative (2.6):

$$DD\Psi = \frac{i}{4}R_{\alpha\beta}\,\widehat{\sigma}^{\alpha\beta}\Psi. \tag{3.4}$$

One can immediately verify that equation (3.1) can be derived from the Lagrange four-form

$$L_{D^2} = L^{(c)} + L^{(p)}, \tag{3.5}$$

where the *convective Lagrangian* and the *polarizational Lagrangian* read, respectively:

$$L^{(c)} := \frac{1}{2}\left(\frac{\hbar^2}{mc}\,{}^*\overline{D\Psi}\wedge D\Psi + {}^*mc\,\overline{\Psi}\,\Psi\right), \tag{3.6}$$

$$L^{(p)} \;=\; M_{\alpha\beta}\wedge R^{\alpha\beta} + M_\alpha\wedge T^\alpha - \frac{\hbar^2}{8mc}\,T\wedge{}^*T\,\overline{\Psi}\,\Psi. \tag{3.7}$$

Here $M_{\alpha\beta}$ is the *Lorentz gravitational moment* given by

$$M_{\alpha\beta} := \frac{\hbar^2}{16mc}\overline{\Psi}({}^*\widehat{\sigma}\widehat{\sigma}_{\alpha\beta} + \widehat{\sigma}_{\alpha\beta}{}^*\widehat{\sigma})\Psi, \tag{3.8}$$

and

$$M_\alpha := \frac{\hbar^2}{4mc}\Big[i\,\overline{\Psi}\,{}^*\widehat{\sigma}D_\alpha\Psi - i\,D_\alpha\overline{\Psi}{}^*\widehat{\sigma}\,\Psi - (e_\alpha\rfloor{}^*\overline{D\Psi})\,\Psi - \overline{\Psi}(e_\alpha\rfloor{}^*D\Psi)\Big]. \tag{3.9}$$

We are now in a position to find the decosmposition of the energy-momentum and spin currents into the convective and polarization parts corresponding to the decomposition of the Lagrangian (3.5),

$$\Sigma_\alpha = \Sigma_\alpha^{(c)} + \Sigma_\alpha^{(p)}, \tag{3.10}$$

$$\tau_{\alpha\beta} = \tau_{\alpha\beta}^{(c)} + \tau_{\alpha\beta}^{(p)}. \tag{3.11}$$

A straightforward use of the *convective* Lagrangian (3.6) in (2.9) and (2.10) yields the the three-forms

$$\begin{aligned}\Sigma_\alpha^{(c)} := \frac{mc}{2}\,\overline{\Psi}\Psi\,\eta_\alpha + \frac{\hbar^2}{4mc}\Big[&{}^*(\overline{D\Psi})D_\alpha\Psi + D_\alpha\overline{\Psi}{}^*D\Psi \\ &+(e_\alpha\rfloor{}^*\overline{D\Psi})\wedge D\Psi + \overline{D\Psi}\wedge(e_\alpha\rfloor{}^*D\Psi)\Big],\end{aligned} \tag{3.12}$$

$$\tau_{\alpha\beta}^{(c)} = -\frac{i\hbar^2}{8mc}\left({}^*\overline{D\Psi}\widehat{\sigma}_{\alpha\beta}\Psi - \overline{\Psi}\widehat{\sigma}_{\alpha\beta}{}^*D\Psi\right). \tag{3.13}$$

For the *polarizational* part, we need the derivatives with respect to curvature and torsion:

$$\frac{\partial L^{(p)}}{\partial R^{\alpha\beta}} = M_{\alpha\beta}, \tag{3.14}$$

$$\frac{\partial L^{(p)}}{\partial T^\alpha} = M_\alpha + \frac{\hbar^2}{4mc}\,(e_\alpha\rfloor{}^*T)\,\overline{\Psi}\,\Psi \;=: \check{M}_\alpha. \tag{3.15}$$

Here we recover the correct *translational grativational moment* [3]

$$\check{M}_\alpha = -\frac{i\hbar^2}{4mc}\,[\overline{\Psi}\,(e_\alpha\rfloor^*\,\widehat{\sigma})\wedge D\Psi + \overline{D\Psi}\wedge(e_\alpha\rfloor^*\,\widehat{\sigma})\,\Psi]. \tag{3.16}$$

In (3.15) we used the Riemann-Cartan the identity which holds for all spinor fields satisfying the Dirac equation:

$$i\left(\overline{\Psi}\,{}^*\widehat{\sigma}\wedge D\Psi-\overline{D\Psi}\wedge{}^*\widehat{\sigma}\Psi\right)={}^*D(\overline{\Psi}\Psi)-{}^*T\,\overline{\Psi}\Psi. \tag{3.17}$$

Substituting (3.14),(3.15) into (2.9),(2.10), we finally obtain the polarizational energy-momentum and spin currents of a Dirac particle in the Riemann-Cartan spacetime:

$$\Sigma_\alpha^{(p)}=D\,\check{M}_\alpha+\Sigma_\alpha^{(RC)}, \tag{3.18}$$

$$\tau_{\alpha\beta}^{(p)}=\vartheta_{[\alpha}\wedge\check{M}_{\beta]}+D\,M_{\alpha\beta}+\tau_{\alpha\beta}^{(RC)}. \tag{3.19}$$

The terms $\Sigma_\alpha^{(RC)}$ and $\tau_{\alpha\beta}^{(RC)}$ contain curvature and torsion explicitly; these contributions are absent in flat Minkowski spacetime. For completeness, we give their explicit form:

$$\begin{aligned}\Sigma_\alpha^{(RC)}=&\,(e_\alpha\rfloor M_{\rho\sigma})\wedge R^{\rho\sigma}-\check{M}_\alpha\wedge T+(e_\alpha\rfloor\check{M}_\beta-e_\beta\rfloor\check{M}_\alpha)\wedge T^\beta\\&-(e_\alpha\rfloor e_\beta\rfloor U)\wedge T^\beta+(e_\alpha\rfloor U)\wedge T+(e_\alpha\rfloor{}^*U)\wedge{}^*T\\&-\frac{\hbar^2}{8mc}\,[(e_\alpha\rfloor T)\,{}^*T+T\wedge e_\alpha\rfloor{}^*T]\,\overline{\Psi}\Psi,\end{aligned} \tag{3.20}$$

$$\tau_{\alpha\beta}^{(RC)}=-e_\gamma\rfloor(M_{\alpha\beta}\wedge T^\gamma). \tag{3.21}$$

As was noticed in [3], the three-form

$$U=\frac{1}{2}\,\vartheta^\alpha\wedge\check{M}_\alpha\;=-\,\frac{i\hbar^2}{4mc}\left(\overline{\Psi}\,{}^*\widehat{\sigma}\wedge D\Psi-\overline{D\Psi}\wedge{}^*\widehat{\sigma}\Psi\right) \tag{3.22}$$

plays a role of a "superpotential" from which all the moments of a Dirac particle are generated. For comparison, it is instructive to collect the properties of the moments in a Table I.

TABLE I. Gauge couplings and moments of the Dirac particle

Gauge model		*Generator*	*Field*	*Moment 2-form*	*Dim*
Maxwell	$U(1)$	i	F	$\frac{e\hbar}{2mc}\overline{\Psi}{}^*\widehat{\sigma}\Psi$	$[e]$
Yang-Mills	$SU(N)$	τ_K	F^K	$-i\frac{e\hbar}{2mc}\overline{\Psi}\tau_K{}^*\widehat{\sigma}\Psi$	$[e]$
Poincaré gravity	T_4	D_α	T^α	$-i\frac{\hbar^2}{4mc}\{\overline{\Psi}(e_\alpha\rfloor{}^*\widehat{\sigma})\wedge D\Psi$ $+\overline{D\Psi}\wedge(e_\alpha\rfloor{}^*\widehat{\sigma})\Psi\}$	$[mc]$
	$SO(1,3)$	$\frac{i}{4}\widehat{\sigma}_{\alpha\beta}$	$R^{\alpha\beta}$	$\frac{\hbar^2}{16mc}\overline{\Psi}({}^*\widehat{\sigma}\widehat{\sigma}_{\alpha\beta}+\widehat{\sigma}_{\alpha\beta}{}^*\widehat{\sigma})\Psi$	$[\hbar]$

IV. DISCUSSION AND CONCLUSION

In this paper, we have generalized our previous discussion [3] of the gravitational moments to the case of the Riemann-Cartan spacetime.

Although the structure of the torsion- and curvature-dependent terms (3.20) and (3.21) is not physically transparent, it is remarkable that the gravitational moments $\check{M}_\alpha$ and $M_{\alpha\beta}$ are the same for a Minkowski and a Riemann-Cartan spacetime. Moreover, the specific coupling $\{moment \times gauge\ field\ strength\}$ on the Lagrangian level is correctly reproduced in the Poincaré gauge–Dirac theory, see (3.7).

The results obtained are helpful for deepening our understanding of the classical limit of the Dirac theory. A relevant general discussion of the low-energy limit can be found in [8] (for non-inertial frames in flat spacetime), and in [9] (for the Riemann-Cartan spacetime).

The present study clearly demonstrates the complete consistency of the properties of *two* gravitational moments with the fundamental structure of the Poincaré gauge gravity which naturally operates with the two types of gravitational charge, mass and spin. At the same time, the definition and properties of a gravitational moment in a purely Riemannian spacetime of Einstein's general relativity (GR) remain unclear. A somewhat paradoxical situation arises: only a *translational* gravitational charge (mass, or energy-momentum) is available in GR, but evidently only a *Lorentz* gravitational moment can survive for the case of the vanishing torsion [recall that a moment is "conjugated" to a corresponding gauge field strength, cf. (3.14)-(3.15)]. This problem will be analyzed elsewhere.

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